Assignment 6

This homework is due *Friday*, November 4.

There are total 40 points in this assignment. 36 points is considered 100%. If you go over 36 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 4.1–4.3 in O'Neill.

(1) [2pt] (4.1.1) None of the following are surfaces. At which points p is it impossible to find a proper patch in M that will cover a neighborhood of p in M? (Sketch/some sort of explanation only — formal proof not required.)
  (a) [2pt] Cone $M:\ z^2 = x^2 + y^2$.
  (b) [2pt] Closed disk $M:\ x^2 + y^2 \leq 1, z = 0$.
  (c) [2pt] Folded plane $M:\ xy = 0, x \geq 0, y \geq 0$.

(2) (4.1.4) In which of the following cases is the mapping $x : \mathbb{R}^2 \to \mathbb{R}^3$ a patch?
  (a) [3pt] $x(u, v) = (u, uv, v)$.
  (b) [3pt] $x(u, v) = (u, u^2, v + v^3)$.
  (Recall that x is one-to-one if and only if $x(u, v) = x(u_1, v_1)$ implies $(u, v) = (u_1, v_1)$.)

(3) (4.1.5)
  (a) [3pt] Prove that $M:\ (x^2 + y^2)^2 + 3z^2 = 1$ is a surface.
  (b) [3pt] For which values of c is $M:\ z(z - 2) + xy = c$ a surface?

(4) (a) [3pt] Determine the intersection of the xy plane $z = 0$ and the *saddle* surface

$$M:\ z = f(x, y), \quad f(x, y) = y^2 - x^2.$$

  On which regions of the plane is $f > 0$? $f < 0$? How does this surface get its name?
  (b) [3pt] (4.1.6) Determine the intersection of the xy plane $z = 0$ and the *monkey saddle*

$$M:\ z = f(x, y), \quad f(x, y) = y^3 - 3yx^2.$$

  On which regions of the plane is $f > 0$? $f < 0$? How does this surface get its name?

(5) [4pt] For the one-sheet hyperboloid $M:\ x^2 + y^2 - z^2 = 1$, find two ruled parametrizations. (*Hint:* Start by assuming that the base curve β in $\beta(u) + v\delta(u)$ is the circle $z = 0$, i.e. $(\cos u, \sin u, 0)$. Put $\delta = (a(u), b(u), c(u))$ and find a, b, c.)

— *see next page* —

(6) (4.3.1) Let $\mathbf{x}$ is the geographical patch on the sphere $\Sigma: \; p_1^2 + p_2^2 + p_3^2 = 1$. Find the coordinate expression $f(\mathbf{x})$ for the following functions on Σ:
 (a) [2pt] $f(p) = p_1^2 + p_2^2$.
 (b) [2pt] $f(p) = (p_1 - p_2)^2 + p_3^2$.

(7) [4pt] (4.3.5a) Prove that $v = (v_1, v_2, v_3)$ is tangent to $M: \; z = f(x, y)$ at a point p of M if and only if

$$v_3 = \frac{\partial f}{\partial x}(p_1, p_2)v_1 + \frac{\partial f}{\partial y}(p_1, p_2)v_2.$$

(*Hint:* For Monge patch $\mathbf{x}$, find $\mathbf{x}_u$ and $\mathbf{x}_v$.)

(8) [4pt] (4.3.8) Find a non-vanishing normal vector field on $M: \; z = xy$ and two tangent vector fields that are linearly independent at each point. (*Hint:* If you want to find tangent vectors first, start by finding convenient tangent vectors at each point. Then take vector orthogonal to them, for example, cross product. If you want to find normal vector first, use general fact for level surfaces.)